Solution to Problems –11

Problem A: *Let* $(a_n)_{n=1}^{\infty}$ *be a sequence of positive real numbers with all terms different from 1. Show that if* $\lim_{n\to\infty} a_n = 1$*, then*

$$\lim_{n\to\infty} \frac{\ln(a_n)}{a_n - 1} = 1.$$

Answer: First we note that

($\circledast$) for all $x > -1$ we have $\frac{x}{x+1} \le \ln(1+x) \le x$.

[Why is ($\circledast$) true? Consider functions $f, g, h : (-1, \infty) \longrightarrow \mathbb{R}$ defined by

$$f(x) = \frac{x}{x+1}, \quad g(x) = \ln(1+x) \quad \text{and} \quad h(x) = x \quad \text{for } x > -1.$$

Clearly, $f(0) = g(0) = h(0) = 0$ and for each $x > -1$ we have

$$f'(x) = \frac{1}{(x+1)^2}, \quad g'(x) = \frac{1}{1+x}, \quad \text{and} \quad h'(x) = 1.$$

Hence, if $x \ge 0$ then $f'(x) \le g'(x) \le h'(x)$ and if $-1 < x < 0$ then $f'(x) > g'(x) > h'(x)$. Consequently, as $f(0) = g(0) = h(0) = 0$, $f(x) \le g(x) \le h(x)$ for all $x > -1$.]

Since $\lim_{n\to\infty} a_n = 1 > 0$, for sufficiently large n we have $a_n > 0$. Without loss of generality $a_n > 0$ for all n and thus $a_n - 1 > -1$ for all n. It follows from ($\circledast$) that, for all n,

$$\frac{a_n - 1}{1 + (a_n - 1)} \le \ln\big(1 + (a_n - 1)\big) = \ln(a_n) \le a_n - 1.$$

Hence, if $a_n > 1$ then

$$\frac{1}{a_n} \le \frac{\ln(a_n)}{a_n - 1} \le 1$$

and therefore by the squeeze principle, the subsequence of $\left(\frac{\ln(a_n)}{a_n-1}\right)_{n=1}^{\infty}$ consisting of those terms for which $a_n - 1 > 0$, if infinite, converges to 1. Similarly, if $a_n < 1$ then

$$\frac{1}{a_n} \ge \frac{\ln(a_n)}{a_n - 1} \ge 1,$$

and therefore by the squeeze principle, the subsequence of $\left(\frac{\ln(a_n)}{a_n-1}\right)_{n=1}^{\infty}$ consisting of those terms for which $a_n - 1 < 0$, if infinite, converges to 1. Consequently, $\lim_{n\to\infty} \frac{\ln(a_n)}{a_n-1} = 1$.

No correct solution were received

NOTE: my biggest apologies, but I made a typo in the Problem B, making the statement not true. The correct problem (with solution) are given below. –Andrzej Rosłanowski.

Problem B: *Let* $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ *be sequences of positive real numbers such that*

$$\lim_{n\to\infty} a_n^n = a > 0 \quad \text{and} \quad \lim_{n\to\infty} b_n^n = b > 0.$$

Suppose that $p, q > 0$ *satisfy* $p + q = 1$*. Prove that*

$$\lim_{n\to\infty} (pa_n + qb_n)^n = a^p b^q$$

Answer: Note first that if $\lim\limits_{n\to\infty} a_n^n = a > 0$ then $\lim\limits_{n\to\infty} a_n = 1$. Assume now that the terms of the sequences $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ are different from 1. By Problem A, we have then

$$(\spadesuit) \qquad\qquad \lim_{n\to\infty} \frac{n\ln(a_n)}{n(a_n-1)} = 1.$$

But the assumtion $\lim\limits_{n\to\infty} a_n^n = a > 0$ and the continuity of the logarithm function imply that $\lim\limits_{n\to\infty} n\ln(a_n) = \ln(a)$. Consequently, $(\spadesuit)$ gives us

$$\lim_{n\to\infty} n(a_n-1) = \lim_{n\to\infty} n\ln(a_n) = \ln(a).$$

Note that the above equalities hold also for the subsequence consisting of the terms equal 1, if it is infinite. Therefore $\lim\limits_{n\to\infty} n(a_n-1) = \ln(a)$ without the additional assumption on a_n, and similarly also $\lim\limits_{n\to\infty} n(b_n-1) = \ln(b)$.

Finally,

$$\lim_{n\to\infty} n\ln\big(pa_n + qb_n\big) = \lim_{n\to\infty} n\big(p(a_n-1) + q(b_n-1)\big) = \ln\big(a^p b^q\big).$$

(Above we use again Problem A for $c_n = pa_n + qb_n$.)

Correct solution was received from :

(1) Cody Anderson POW 11B: ♡

Problem C: *Find the limit of the sequence* $(a_n)_{n=1}^{\infty}$*, where*

$$a_n = \left(1+\frac{1}{n^2}\right)\cdot\left(1+\frac{2}{n^2}\right)\cdot\ldots\cdot\left(1+\frac{n}{n^2}\right), \qquad \textit{for } n=1,2,3,\ldots$$

Answer: First we note that

(♣) for all $x>0$ we have $x-\frac{x^2}{2}<\ln(1+x)<x$.

[Why is (♣) true? Consider functions $f,g,h:(-1,\infty)\longrightarrow\mathbb{R}$ defined by

$$f(x)=x-\frac{x^2}{2},\quad g(x)=\ln(1+x)\quad\text{and}\quad h(x)=x\quad\text{for } x>-1.$$

Clearly, $f(0)=g(0)=h(0)=0$ and for each $x>0$ we have

$$f'(x)=1-x<g'(x)=\frac{1}{1+x}<h'(x)=1.$$

Consequently, as $f(x)<g(x)<h(x)$ for all $x>0$.]

Let

$$b_n=\ln(a_n)=\sum_{k=1}^{n}\ln\left(1+\frac{k}{n^2}\right).$$

By (♣) we have

$$\frac{k}{n^2}-\frac{k^2}{2n^4}<\ln\left(1+\frac{k}{n^2}\right)<\frac{k}{n^2}.$$

It is well known that

$$\sum_{k=1}^{n}k=\frac{n(n+1)}{2}\quad\text{and}\quad\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$$

(easy inductive proofs). Therefore

$$\frac{n(n+1)}{2n^2}-\frac{n(n+1)(2n+1)}{12n^4}<b_n<\frac{n(n+1)}{2n^2},$$

and hence $\lim\limits_{n\to\infty}b_n=\frac{1}{2}$. Consequently,

$$\lim_{n\to\infty}a_n=\sqrt{e}.$$

Correct solution was received from :

(1) Cody Anderson — POW 11C: ♡